SIRP.T.SCIENCECOLLEGE,MODASA

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Certificate

This is to certify that the following students of B.Sc.(Sem-IV) has successfully completed the project entitled **Study of Connecteness on Topological Space** under the guidance of Dr. V. R. Patel, Head and Assistant Professor, Department of Mathematics, SIR P. T. SCIENCE COLLEGE, MODASA during year 2022-2023.

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Theorem 10:- Let $(Y, \mathcal{T}_Y)$ be sub-space of a topological space $(X, \mathcal{T})$. Let $\mathcal{B}$ be a base for $\mathcal{T}$, then $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$ is a base for $\mathcal{T}_Y$.

Proof:- Let H be any $\mathcal{T}_Y$-open set and let $x \in H$. Then $\exists$ a $\mathcal{T}$-open set G such that $H = G \cap Y$.

Now, G is a $\mathcal{T}$-open set containing x & $\mathcal{B}$ is base for $\mathcal{T}$. So, $\exists$ a set B in $\mathcal{B}$ such that

$$x \in B \subseteq G$$

$$x \in B \cap Y \subseteq G \cap Y = H$$

$\therefore$ Thus, to each $H \in \mathcal{T}_Y$ & $x \in H$ $\exists$ $B \cap Y \in \mathcal{B}_Y$ such that

$$x \in B \cap Y \subseteq H.$$

This shows that $\mathcal{B}_Y$ is a base for $\mathcal{T}_Y$.

Exercise 6

1 Let $X = \{a, b, c, d, e\}$
and let $\mathcal{T} = \{X, \phi, \{a\}, \{a, b\}, \{a, c, d\}, \{a, b, c, d\}\}$ be a topology on X.
If $Y = \{a, d, e\}$, find $\mathcal{T}$-relative topology for Y.

2. Show that every subspace of a discrete space is discrete

Hint: Let $(Y, \mathcal{T}_Y)$ be a subspace of a discrete space $(X, \mathcal{T})$.
Then, for each $y \in Y$, we have
$\{y\} = \{y\} \cap Y$, where $\{y\} \in \mathcal{T}$.

$\Rightarrow A \cap Y$ is a $\mathcal{J}_Y$-nhd of y

$\Rightarrow A$ is a $\mathcal{J}_Y$-nhd of y

$[\because A \subset Y \Rightarrow A \cap Y = A]$

$\Rightarrow y \in \mathcal{J}_Y\text{-int}(A)$

sub-spaces

$\therefore \mathcal{J}_Y\text{-int}(A) \supset \mathcal{J}\text{-int}(A)$.

Remark. In general, $\mathcal{J}_Y\text{-int}(A) \neq \mathcal{J}\text{-int}(A)$.

Ex Let $X = \{a, b, c, d, e\}$

and let $\mathcal{J} = \{X, \emptyset, \{a\}, \{a,b\}, \{a,c,d\}, \{a,b,e\}, \{a,b,c,d\}\}$ be a topology on X.

Let $Y = \{a, c, e\}$.

Then, $\mathcal{J}_Y = \{Y, \emptyset, \{a\}, \{a,c\}, \{a,e\}\}$.

Now, if $A = \{a,c\} \subset Y$, then clearly $\mathcal{J}_Y\text{-int}(A) = \{a,c\}$ and $\mathcal{J}\text{-int}(A) = \{a\}$.

Thus, in general, $\mathcal{J}_Y\text{-int}(A) \neq \mathcal{J}\text{-int}(A)$.

Theorem 9:- Let $(Y, \mathcal{J}_Y)$ be a subspace of a topological $(X, \mathcal{J})$ and let $A \subset Y$. Then, $\mathcal{J}_Y\text{-bd}(A) \subset \mathcal{J}\text{-bd}(A)$

Proof:- $y \in \mathcal{J}_Y\text{-bd}(A) \Rightarrow y \in cl_Y(A) \cap cl_Y(Y-A)$

$\Rightarrow y \in cl_Y(A)$ & $y \in cl_Y(Y-A)$

$\Rightarrow y \in cl_X(A) \cap Y$ & $y \in \{cl_X(Y-A)\} \cap Y$

$\Rightarrow y \in cl_X(A)$ & $y \in cl_X(X-A)$

$[\because \overline{Y-A} \subseteq \overline{X-A}]$

$\Rightarrow y \in \mathcal{J}\text{-bd}(A)$.

$\therefore \mathcal{J}_Y\text{-bd}(A) \subset \mathcal{J}\text{-bd}(A)$.

Th. 6 :- Let $(Y, \mathfrak{J}_Y)$ be a subspace of a topological space. Let $A \subset Y$. Then $cl_Y(A) = cl_X(A) \cap Y$.

Proof :- since $cl_X(A)$ is $\mathfrak{J}$-closed, it follows that $cl_X(A) \cap Y$ is $\mathfrak{J}_Y$-closed, Thus, $cl_X(A) \cap Y$ is a $\mathfrak{J}_Y$-closed superset of A but, $cl_Y(A)$ being the smallest $\mathfrak{J}_Y$-closed superset of A.

$\therefore cl_Y(A) \subseteq cl_X(A) \cap Y$ —— ①

Again, $cl_Y(A)$ being $\mathfrak{J}_Y$-closed, we have

$cl_Y(A) = F \cap Y$ for some $\mathfrak{J}$-closed set F.

$\therefore A \subseteq cl_Y(A) = F \cap Y$ & so $A \subseteq F$.

Now, $A \subseteq F \Rightarrow cl_X(A) \subseteq \bar{F} = F$ [$\because$ F is $\mathfrak{J}$-closed]

$\therefore cl_X[A] \cap Y \subseteq F \cap Y = cl_Y(A)$ —— ②

Hence from ① & ② we have, $cl_Y(A) = cl_X(A) \cap Y$

Theorem 7 :- Let $(Y, \mathfrak{J}_Y)$ be a subspace of a topological space $(X, \mathfrak{J})$. Let $A \subset Y$. Then a point $y \in Y$ is a $\mathfrak{J}_Y$-limit point of A if and only if y is a $\mathfrak{J}$-limit point of A.

Proof :- y is a $\mathfrak{J}_Y$-limit point of A

$\Leftrightarrow [M - \{y\}] \cap A \neq \emptyset \ \forall \ \mathfrak{J}_Y$-nhds M of y.

$\Leftrightarrow [(N \cap Y) - \{y\}] \cap A \neq \emptyset \ \forall \ \mathfrak{J}$-nhd. N of y.

$\Leftrightarrow (N - \{y\}) \cap A \neq \emptyset \ \forall \ \mathfrak{J}$-nhd. N of y.

$\Leftrightarrow$ y is a $\mathfrak{J}$-limit point of A

Remark :- If $D_Y(A)$ and $D_X(A)$ denote the derived sets of A in $(Y, \mathfrak{J}_Y)$ & $(X, \mathfrak{J})$ respectively, then

$D_Y(A) = D_X(A) \cap Y$.

Theorem :- 8. Let $(Y, \mathfrak{J}_Y)$ be a subspace of topological space $(X, \mathfrak{J})$. Let $A \subset Y$. Then $\mathfrak{J}_Y\text{-int}(A) \supset \mathfrak{J}\text{-int}(A)$.

Proof :- $y \in \mathfrak{J}\text{-int}(A) \Rightarrow y$ is a $\mathfrak{J}$-interior point of A

$\Rightarrow$ A is a $\mathfrak{J}$-nhd. of y

Page :
Date :

$= G \cap (Y \cap Z) = G \cap Z \quad [\because Z \subset Y]$

Thus, $E = G \cap Z$ for some $G \in \mathcal{J}$ and therefore, $E \in \mathcal{J}_Z$.

So, $E \in \eta \Rightarrow E \in \mathcal{J}_Z$ i.e. $\eta \subseteq \mathcal{J}_Z$

Again, let $W \in \mathcal{J}_Z$. Then $W = V \cap Z$ for some $V \in \mathcal{J}$.

but, $V \in \mathcal{J} \Rightarrow V \cap Y \in \gamma$

$\Rightarrow (V \cap Y) \cap Z \in \eta \; [\because (Z, \eta)$ is a s.p. of $(X, \mathcal{J})]$

$\Rightarrow V \cap (Y \cap Z) = V \cap Z \in \eta$

$\Rightarrow W \in \eta$

Thus, $W \in \mathcal{J}_Z \Rightarrow W \in \eta$ and therefore, $\mathcal{J}_Z \subseteq \eta$.

Hence $\mathcal{J}_Z = \eta$.

Accordingly, (Z, η) is a subspace of $(X, \mathcal{J})$.

Theorem 5 :- Let $(Y, \mathcal{J}_Y)$ be a subspace of a topological space $(X, \mathcal{J})$. Let $y \in Y$. Then a subset M of Y is a $\mathcal{J}_Y$-nhd. of y iff $M = N \cap Y$ for some $\mathcal{J}$-nhd. N of y.

Proof :- Let M be a $\mathcal{J}_Y$-nhd. of y.

Then, $\exists$ a $\mathcal{J}_Y$-open set H such that $y \in H \subseteq M$.

Now, H being $\mathcal{J}_Y$-open, we have $H = G \cap Y$ for some $G \in \mathcal{J}_Z$

$\therefore \; y \in G \cap Y \subseteq M$.

Let $M \cup G = N$.

Then, $y \in G \subset M \cup G = N$, where $G \in \mathcal{J}$.

This shows that N is a $\mathcal{J}$-nhd of y.

Further, $N \cap Y = (M \cup G) \cap Y = (M \cap Y) \cup (G \cap Y) = M$.

Conversely, let $M = N \cap Y$ for some $\mathcal{J}$-nhd N of y.

Then, $\exists$ a $\mathcal{J}$-open set G such that $y \in G \subseteq N$.

Consequently, $y \in G \cap Y \subseteq N \cap Y = M$

This shows that M is a $\mathcal{J}_Y$-nhd of y.

$[\because G \cap Y \in \mathcal{J}_Y]$

$\Leftrightarrow (Y-A) = G \cap Y$ for some $G \in \mathfrak{J}$

$\Leftrightarrow A = Y - (G \cap Y) = (Y - G) \cup (Y - Y)$

$\Leftrightarrow A = (Y - G) = Y \cap G^c$, where G^c is $\mathfrak{J}$-closed

$\Leftrightarrow A = Y \cap F$, where $F = G^c$ is $\mathfrak{J}$-closed

Theorem:- 3. Let $(Y, \mathfrak{J}_Y)$ be a subspace of a topological space $(X, \mathfrak{J})$. Then in order that every $\mathfrak{J}_Y$-open subset of Y be $\mathfrak{J}$-open a necessary and sufficient condition is that Y be $\mathfrak{J}$-open

Proof:- We first suppose that every Y_Y-open subset of Y is $\mathfrak{J}$-open. Then Y being $\mathfrak{J}_Y$-open it follows from the given condition that Y is $\mathfrak{J}$-open

Conversly. Let Y be $\mathfrak{J}$-open let H be any $\mathfrak{J}_Y$-open set

Then, $H = G \cap Y$ for some $G \in \mathfrak{J}$.

but, H being the intersection of two $\mathfrak{J}$-open sets, It is $\mathfrak{J}$-open

Thus, in this case, every $\mathfrak{J}_Y$-open set is $\mathfrak{J}$-open

Theorem 5. Let $(X, \mathfrak{J})$, (Y, r) and (Z, n) be three topological spaces such that (Y, r) is a subspace of $(X, \mathfrak{J})$ and (Z, n) be a subspace of (Y, r) Then (Z, n) is a subspace of $(X, \mathfrak{J})$.

Proof:- Clearly $Y \subset X$ and $Z \subset Y$. So $Z \subset X$. In order to prove that (Z, n) is a subspace of $(X, \mathfrak{J})$, we must show that the $\mathfrak{J}$-relativized topology on Z is n i.e. $\mathfrak{J} = n$.

Let $E \in n$. Then

$E = H \cap Z$ for some $H \in r$ [$\because (Z, n)$ is a subspace of (Y, r)]

$= (G \cap Y) \cap Z$ for some $G \in \mathfrak{J}$

be expressed as the union of singleton subsets of N, each one of which is $\mathcal{U}_N$-open. And, the arbitrary union of sets being open it follows that A is $\mathcal{U}_N$-open.
Thus, every subset of N is $\mathcal{U}_N$-open.
Hence, the relativized topology for N is the discrete topology.

Remark. If $(Y, \mathcal{J}_Y)$ is a subspace of the space $(X, \mathcal{J})$, then a set open in X is not necessarily open in Y.

Ex. Consider the usual topological space $(R, \mathcal{U})$. Let $A = [0,1]$ and let $\mathcal{U}_A$ be the relativized topology on A. Now, if we consider $]0,1]$, then evidently, it is not open in $(R, \mathcal{U})$. But we can write

$$]0,1] =]0,2[\cap [0,1] =]0,2[\cap A$$

$\therefore]0,1] \in \mathcal{U}_A$, since $]0,2[\in \mathcal{U}$.

This shows that $]0,1]$ is open in A but not in R.

Again, if we consider $]0,\frac{1}{2}]$, then it is clear that it is not open in $(R, \mathcal{U})$. Moreover $]0,\frac{1}{2}]$ can not be expressed as the intersection of a $\mathcal{U}$-open set and A. So, $]0,\frac{1}{2}]$ is not $\mathcal{U}_A$-open.

Thus, $]0,\frac{1}{2}]$ is a subset of A, which is neither $\mathcal{U}$-open nor $\mathcal{U}_A$-open.

Theorem 2. Let $(Y, \mathcal{J}_Y)$ be a subspace of a topological space $(X, \mathcal{J})$. Then, a subset A of Y is $\mathcal{J}_Y$-closed if and only if $A = F \cap Y$ for some $\mathcal{J}$-closed subset F of X.

Proof. A is $\mathcal{J}_Y$-closed

$\Leftrightarrow (Y - A)$ is $\mathcal{J}_Y$-open

Remark. This topology $\mathcal{J}_Y$ is the relativized or inherited topology on Y. Also $(Y, \mathcal{J}_Y)$ is called the sub-space of $(X, \mathcal{J})$.

Hereditary Property. A property of a topological space is said to be a hereditary property, if it is satisfied by every subspace of the given space

Illustrative Examples

Ex. 1. Let $X = \{a, b, c, d, e\}$ and let
$$\mathcal{J} = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$$
be a topology on X

Let $Y = \{a, d, e\}$

Then, we have

$X \cap Y = Y$; $\emptyset \cap Y = \emptyset$; $\{a\} \cap Y = \{a\}$; $\{c, d\} \cap Y = \{d\}$

$\{a, c, d\} \cap Y = \{a, d\}$ and $\{b, c, d, e\} \cap Y = \{d, e\}$.

$\therefore$ $\mathcal{J}$ relativized topology on Y is given by
$$\mathcal{J}_Y = \{Y, \emptyset, \{a\}, \{d\}, \{a, d\}, \{d, e\}\}$$

Ex. 2. Consider the usual topological space $(\mathbb{R}, \mathcal{U})$. Let $\mathbb{N}$ be the set of all natural numbers. Then the relativized topology on $\mathbb{N}$ is the discrete topology

Proof:- For an arbitrary $n \in \mathbb{N}$, we have
$$\{n\} =]n - \tfrac{1}{2}, n + \tfrac{1}{2}[\cap \mathbb{N}$$
$$\in \mathcal{U}_N, \text{ since }]n - \tfrac{1}{2}, n + \tfrac{1}{2}[\in \mathcal{U}$$

Thus each singleton subset of $\mathbb{N}$ is $\mathcal{U}_N$-open

Now if A is any subset of $\mathbb{N}$, then it can

Introduction. It is always possible to construct new topologies from the given ones. The simplest one is the relativized topology

If $(X, \mathcal{J})$ is a topological space and $Y \subset X$, then Y can inherit a topology from X as shown in the following result.

Theorem 1. Let $(X, \mathcal{J})$ be a topological space and let $Y \subset X$. Then, the collection $\mathcal{J}_Y = \{G \cap Y : G \in \mathcal{J}\}$ is a topology on Y

Proof :- We observe that $\mathcal{J}_Y$ satisfies the following properties: i) $\phi \in \mathcal{J}$ and $\phi \cap Y = \phi \Rightarrow \phi \in \mathcal{J}_Y$;

ii) $X \in \mathcal{J}$ and ~~$\phi \cap Y = \phi$~~ Y $X \cap Y = Y \Rightarrow Y \in \mathcal{J}_Y$;

iii) Let $\{H_\alpha : \alpha \in \Lambda\}$ be any family of sets in $\mathcal{J}_Y$. Then for each $\alpha \in \Lambda$ $\exists$ a set $G_\alpha \in \mathcal{J}$ such that $H_\alpha = G_\alpha \cap Y$.

$$\therefore \cup\{H_\alpha : \alpha \in \Lambda\} = \cup\{G_\alpha \cap Y : \alpha \in \Lambda\}$$
$$= [\cup\{G_\alpha : \alpha \in \Lambda\}] \cap Y$$
$$\in \mathcal{J}_Y, \text{ since } \cup\{G_\alpha : \alpha \in \Lambda\} \in \mathcal{J};$$

iv) Let H_1 and H_2 be any two sets in $\mathcal{J}_Y$. Then $H_1 = G_1 \cap Y$ and $H_2 = G_2 \cap Y$ for some G_1, G_2 in $\mathcal{J}$

$$\therefore H_1 \cap H_2 = (G_1 \cap Y) \cap (G_2 \cap Y)$$
$$= (G_1 \cap G_2) \cap Y$$
$$\in \mathcal{J}_Y, \text{ since } G_1 \cap G_2 \in \mathcal{J}.$$

Hence $\mathcal{J}_Y$ is a topology for Y.