SIRP.T.SCIENCECOLLEGE,MODASA

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Certificate

This is to certify that the following students of B.Sc.(Sem-IV) has successfully completed the project entitled **Properties of Polynomials in Ring Theory** under the guidance of Dr. K. N. Darji, Assistant Professor, Department of Mathematics, SIR P. T. SCIENCE COLLEGE, MODASA during year 2022-2023.

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Introduction

In Chapter 18, we proved that an integral domain can be embedded into a field. In this Chapter, defining an integral domain $D[x]$ with the help of a given integral domain D, we will show that D can also be embedded into the integral domain $D[x]$. Moreover, we will study important properties of the integral domain $D[x]$. Since we define integral domain $D[x]$ in terms of special sequences in D, it will be convenient to familirize ourselves first with the notion of a sequence and some of the notations associated with it. Once the integral domain $D[x]$ which is familiar to us since our high school days and we will also study the properties of integral domain $D[x]$ in this familiar form.

For a given nonempty set S, a mapping $f: N \to S$ is called a sequence in S (we will consider only infinite sequence here). If for $n \in N$, $f(n) = a_n \in S$, then the sequence f is denoted by $\{a_n\}_{n=1}^{\infty}$ OR simply by $\{x_n\}$. Here x_n is called the nth term of the sequence. Without loss of generality, we can fix the domain of a sequence and if $f(n) = x_n$, then we will denote the sequence by $(x_0, x_1, x_2, \ldots, x_n, \ldots)$.

Suppose f is a sequence in the integral domain D. Then, for the given non-negative integers m and n, $n > m$, the expression $f(m) + f(m+1) + \ldots + f(n)$ is well-defined because of the generalized associative law in D. We will denote this sum by the summation notation. $\sum_{i=m}^{n} f(i)$

EXAMPLE: For sequence $f(i) = i$ in $D = (Z; +; \cdot)$

$$\sum_{i=6}^{11} f(i) = \sum_{i=6}^{11} i = 6+7+8+9+10+11 = 51$$

If $f(i) = (-1)^i$, then

Page

Date

$$\sum_{i=0}^{5} f(i) = (-1)^0 + (-1)^1 + \dots + (-1)^5$$

$$= 1 - 1 + \dots - 1 = 0$$

If $f(i) = 3$, then

$$\sum_{i=2}^{5} f(i) = f(2) + f(3) + f(4) + f(5)$$

$$= 3 + 3 + 3 + 3$$

$$= 12$$

Integral Domain :

A sequence in D having only a finite number of non zero terms is called a polynomial in D.

The set of all polynomials in D is denoted by $D[x]$.

By definition, if $f = (a_0, a_1, \ldots, a_n, \ldots)$ is a polynomial in D, then there exists a non-negetive integer m such that $a_i = 0$ for each $i > m$

Remark :-

(i) We denote the set of all polynomials in D by $D[x]$. In We will associate a special polynomial with the notation x used here.

(ii) Clearly, elements of $D[x]$ being special type of sequences in D, none of the elements of $D[x]$ is an element of D

we will denote elements of $D[x]$ by symbols $f, g, h, \ldots,$ etc.

As mentioned in the beginning of the chapter, we want to make $D[x]$ an integral domain.

For this purpose, we have to first define equality, addition and multiplication in $D[x]$

(i) Equality :

f and g are called equal polynomials (and denoted by $f = g$) if $a_n = b_n$ for each non negetive integer n.

(ii) Addition :-

$f + g = (c_0, c_1, \ldots, c_n, \ldots)$ where $c_n = a_n + b_n$ for each non-negetive integer n.

(iii) multiplication :

$f \cdot g = (d_0, d_1, \ldots, d_n, \ldots)$

where $d_n = a_n b_0 + \ldots + a_0 b_n$

$$= \sum_{i=0}^{n} a_i b_{n-i}$$

$$= \sum_{i+j=n} a_i b_j$$ for each non negetive integer n.

Example :-

$f = (0, 1, 2, 0, 0, 0, \ldots)$ and
$g = (1, 0, -3, 1, 0, 0, 0, \ldots)$ for $f, g \in \mathbb{Z}[x]$
Call the terms after the third in f and all the terms after the ~~for~~ third in f and all the terms after the fourth in g are zero).
By defincation.

$$f + g = (1, 1, -1, 0, 0, 0, \ldots)$$

and

$$f \cdot g = (0, 1, 2, 0, 0, 0, \ldots)(1, 0, -3, 1, 0, 0, 0, \ldots)$$

$$0\cdot(-3) + 1\cdot 0 + 2\cdot 1,\ 0\cdot 1 +$$
$$= (0\cdot 1,\ 0\cdot 0 + 1\cdot 1,\ 0\cdot 0 + 1\cdot 1 + 2\cdot(-3)$$
$$1\cdot(-3) + 2\cdot 0 + 0\cdot 1,\ 0\cdot 0 + 1\cdot 1 + 2\cdot(-3)$$
$$1\cdot 0 + 2\cdot 1 + 0\cdot(-3) + 0\cdot 0$$
$$+ 0\cdot 0 + 0\cdot 1,\ 0\cdot 0 + 1\cdot 0 + 2\cdot 1 + 0\cdot(-3) + 0\cdot 0$$
$$+ 0\cdot 1,\ 0\cdot 0 + 1\cdot 0 + 2\cdot 0 + 0\cdot 1 + 0\cdot(-3) + 0\cdot 0$$
$$+ 0\cdot 1,\ 0, 0, \ldots)$$
$$= (0, 1, 2, -3, -5, 2, 0, 0, 0, \ldots)$$

All the terms after the sixth term are zero in multiplication $f \cdot g$

Remark :-

To obtain simplicity in computation, we will denote the elements of the ring $(Z_n, +_n, \cdot_n)$ simply by $0, 1, 2, \ldots, (n-1)$ instead of $[0], [1], \ldots, [n-1]$, respectively, for $+_n$ and $\cdot_n$.

There is no fear of confusion as the meaning will be clear from the corresponding context.

We define a polynomial with the help of a sequence and hence it is natural to call the terms of a sequence as the terms of a polynomial. Instead of this, they are called the co-efficient of polynomial.

In fact, we have the following definition.

Page

Date

Definition :

If $f = (a_0, a_1, a_2, \ldots, a_n, \ldots)$ is a polynomial in D, then

(i) a_0 is called the constant term of the polynomial f and

(ii) for each positive integer n, a_n is called the nth coefficient of the polynomial f.

Definition :

A non negetive integer p is called the degree of a non zero polynomial $f = (a_0, a_1, \ldots, a_n, \ldots)$ in D if $a_p \neq 0$ and $a_n = 0$ for each $n > p$.

Theorem :-

For nonzero polynomials, $f, g \in D[x]$, $[fg] = [f] + [g]$. That is, the degree of product of two non-zero polynomials is equal to the sum of the degrees of these two polynomials.

Carollary :-

For non zero. polynomials f and $g \in D[x]$, $0 \leq [f] \leq [fg]$

Theorem :-

For an integral domain D, the set $D[x]$ of all polynomials on D is also an integral domain under the binary operations of addition and multiplication.

Theorem :-

The degree of a unit element in $D[x]$ is always zero.

Theorem :-

The integral domain $D[x]$ is not a field

Page
Date

Familiar Form of I.D. $D(x)$

So far, we have studied the Polynomial domain $D[x]$ in its abstract form. Now we want to transform this abstract form to a form with which we are familiar form our school days.

Definition: A Polynomial $f = (a_0, a_1, a_2, \dots) \in D(x)$ is called a constant Polynomial if $a_n = 0$ for each $n \geq 1$

Theorem: $D \cong D'$

Theorem: If $x = (0, 1, 0, \dots)$ Then
$$x^n = (a_0, a_1, a_2, \dots)$$

Theorem: If the degree of the Polynomial $f = (a_0, a_1, a_2, \dots) \in D(x)$ is
$$f = a_0 + a_1 x + \dots + a_m x^m$$
where
$$x = (0, 1, 0, \dots)$$
$$a = (a, 0, 0, \dots) \text{ for each } a \in D.$$

Example: suppose $f = (2, 5, -2, 4, 1, 0, 3, 0, 0, \ldots) \in Z(x)$

Example: If $f = (1, -2, 0, 3, 0, \ldots)$ and

$g = (2, 0, -3, 0, 4, 0, \ldots) \in Z(x)$