SIRP.T.SCIENCECOLLEGE,MODASA

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Certificate

This is to certify that the following students of B.Sc.(Sem-IV) has successfully completed the project entitled **Basic properties of Topological Spaces** under the guidance of Dr. V. R. Patel, Head and Assistant Professor, Department of Mathematics, SIR P. T. SCIENCE COLLEGE, MODASA during year 2022-2023.

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Topology :→

A topology is a family $\mathcal{J}$ of sets which satisfies the two condition:-

① The intersection of any two members of $\mathcal{J}$ is a member of $\mathcal{J}$.

② The union of the members of each subfamily of $\mathcal{J}$ is a member of $\mathcal{J}$.

The set $X = \bigcup \{U : U \in \mathcal{J}\}$ is a necessarily a member of $\mathcal{J}$ because $\mathcal{J}$ is a subfamily of itself and every member of $\mathcal{J}$ is a subset of X.
The set X is called the space of the topology $\mathcal{J}$ and $\mathcal{J}$ is a topology for X.

The pair $(X, \mathcal{J})$ is a topological space.

<u>OR</u>

What is a topology?

Let X be a non-empty set. A topology on X is a collection of open subsets $(\mathcal{J})$ of X. which satisfies the following conditions:-

① $X, \phi \in J$

② The union of the members of each subfamily of J is a member of J

③ The intersection of any two members of J is a member of J.

Example of topology :-

Let the set $X = \{a, b, c\}$ and $J = \{\phi, \{a\}, \{a,b\}, X\}$ then show that J forms a topology on X.

:- First we prove three (3) condition of topology

1> $X, \phi \in J$

2> The union of any elements of T is also a member of T

$\rightarrow \{a\} \cup \{a,b\} = \{a,b\} \in T$

3> The finite intersection of any member of J is also a member of J

$\rightarrow \{a\} \cap \{a,b\} = \{a\} \in J$

Therefore J is a topology on X.

Example : Let $X \{a,b,c,d,e\}$. consider the follo-
-wing classes of subsets of X.

$$S_1 = \{\phi, X, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d,e\}\}$$
$$S_2 = \{\phi, X, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d\}\}$$
$$S_3 = \{\phi, X, \{a\}, \{c,d\}, \{a,c,d\}, \{a,b,d,e\}\}$$

we observe that S_1 is a topology on X since it satisfies the three axioms
But S_2 is not a topology on X since the union $\{a,c,d\} \cup \{b,c,d\} = \{a,b,c,d\}$ of two members of S_2 is not in S_2 and so S_2 doesnot satisfy the third axiom.

similarly it can be seen that S_3 is not a topology on X since the intersection.

$$\{a,c,d\} \cap \{a,b,d,e\} = \{a,d\}$$

of two sets in S_3 / does not belongs to S_3 does not satisfy the second axiom.

Example : A metric space is a special kind of topologyical space. The openysul the axioms $(O_1)-(O_3)$ hold since

(O_1) ϕ and X in a metricspace (X,d) are open

(O_2) The intersection of any finite numbers of open sets in (X,d) is open.

(O_3) Any union (coutable or not) of open sets in (X,d) is open.

This topology defined on metric space is called usual topology on a metric space.

Example: Let $\mathcal{D}$ denote the class of all subsets of X. Then $\mathcal{D}$ satisfies all axioms for a topology on X. This topology is called Discrete topology and $(X, \mathcal{D})$ is called Discrete topological space or simply a Discrete space.

Example: Let X be any infinite set and T be the family consisting of ϕ and complements of finite subsets of X. Show that T is a topology on X.

Analysis: Let X be an infinite set and T be the family consisting of ϕ and subset of X whose complements in X are finite.

To prove that T is a topology on X.

1. $\phi \in T$ and since ϕ is finite $\Rightarrow \phi^c = X \in T$.
2. Let $G_1, G_2 \in T \Rightarrow G_1 \cap G_2 \in T$

If $G_1 \cap G_2 = \phi$ then $G_1 \cap G_2 \in T$

If $G_1 \cap G_2 \neq \phi$, then $G_1 \neq \phi$, $G_2 \neq \phi$.

and $G_1, G_2 \in T \Rightarrow X - G_1$ and $X - G_2$ are finite

$\Rightarrow (X - G_1) \cap (X - G_2)$ is finite.

$\Rightarrow X - (G_1 \cap G_2)$ is finite

$\Rightarrow G_1 \cap G_2 \in T$

Thus in either case $G_1, G_2 \in T \to G_1 \cap G_2 \in T$

(3) Let $G_\alpha \in T \Rightarrow X - G_\alpha$ is finite

$\Rightarrow \bigcap_\alpha (X - G_\alpha)$ is finite

$\Rightarrow \bigcap_\alpha G_\alpha^c$ is finite

$\Rightarrow X - (\bigcap_\alpha G_\alpha^c)$ is finite $\in T$

$\Rightarrow \bigcup_\alpha (G_\alpha^c)^c \in T$

$\Rightarrow \bigcup_\alpha G_\alpha \in T$

Hence all the axioms for a topology are the satisfied

$\Rightarrow$ T is a topology on X

Example : Let X be a nonempty set. The family $I = \{\phi, X\}$ consisting of ϕ and X is itself a topology on X and is called the indiscrete topology or simply an Indiscrete space. It is the Coarsest topology.

Remark :→ when X is a singleton, then the two topologies discrete and indiscrete conicide.

Theorem: The intersection $T_1 \cap T_2$ any two topologies T_1 and T_2 on X is also a topology on X.

Proof:- since T_1 and T_2 are topologies on X, therefore $\phi, X \in T$ and $\phi, X \in T_2$

$$\Rightarrow \phi, X \in T_1 \cap T_2$$

This is $T_1 \cap T_2$ satisfies first axiom for a topology

Also if $G, H \in T_1 \cap T_2$, then
$G, H \in T_1$ and $G, H \in T_2$

since T_1 and T_2 are topologies

$$\Rightarrow G \cap H \in T_1 \text{ and } G \cap H \in T_2$$
$$\Rightarrow G \cap H \in T_1 \cap T_2$$

i.e $T_1 \cap T_2$ satisfies the second axiom for a topology.

Further, Let $G_\alpha \in T_1 \cap T_2$ for every $\alpha \in S$, where S is an arbitrary set
Then $G_\alpha \in T_1$ and $G_\alpha \in T_2$ for every $\alpha \in S$.
but T_1 and T_2 are topologies
$\Rightarrow \bigcup_\alpha G_\alpha \in T_1$ and $\bigcup_\alpha G_\alpha \in T_2 \Rightarrow \bigcup_\alpha G_\alpha \in T_1 \cap T_2$.

i.e $T_1 \cap T_2$ satisfies third axiom for a topology. → Hence the result follows.

Topological Spaces.

Definition :- Let X be a non empty set and $\mathcal{J}$ a collections of subsets of X such that

1. $X \in \mathcal{J}$
2. $\phi \in \mathcal{J}$
3. If $O_1, O_2, \ldots, O_n \in \mathcal{J}$ then $O_1 \cap O_2 \cap \ldots \cap O_n \in \mathcal{J}$
4. If for each $\alpha \in I$, $O_\alpha \in \mathcal{J}$ then $\bigcup_{\alpha \in I} O_\alpha \in \mathcal{J}$.

=> The pair of objects $(X, \mathcal{J})$ is called a topological spaces.

=> Let (X, d) be a metric space. The collection $\mathcal{J}$ of open sets of this metric satisfies the conditions 1, 2, 3 and 4. Thus, (X, d) gives rise to the topological space $(X, \mathcal{J})$.

Definition :- Let (X, d) be a metric space. Let $\mathcal{J}$ be the collection of open set of this metric space. The topological space $(X, \mathcal{J})$ is called the topological space associated with the metric space (X, d) and the metric space (X, d) is said to be give rise to the topological space $(X, \mathcal{J})$.

=> To verify that $(X, \mathcal{J})$ is topological space, one verifies that the specified collection of subjects, $\mathcal{J}$ is a topology; that is, that $\mathcal{J}$ is satisfies 1, 2, 3 and 4.

=> For example, let X and $\mathcal{J}$ be as in Example 6. Then $X \in \mathcal{J}$, for its complement $\phi = c(X)$ is certainly finite. Also $\phi \in \mathcal{J}$, since $c(\phi) = X$.

=> Thus, $\mathcal{J}$ satisfies conditions 1 and 2. Next let $O_1, O_2, \ldots, O_n$ be subsets of X, each of whose complements is finite or all of X. To show that $O_1 \cap O_2 \cap, \ldots, \cap O_n \in \mathcal{J}$ we must show that $c(O_1 \cap O_2 \cap, \ldots, \cap O_n) = c(O_1) \cup c(O_2) \cup, \ldots, \cup c(O_n)$.

=> Either this set is a union of finite sets and hence finite, or for some i, $c(O_i) = X$ and the union is all of X. Finally, for each $\alpha \in I$, let $O_\alpha \in \mathcal{J}$, so that $c(O_\alpha)$ is either finite, or X.

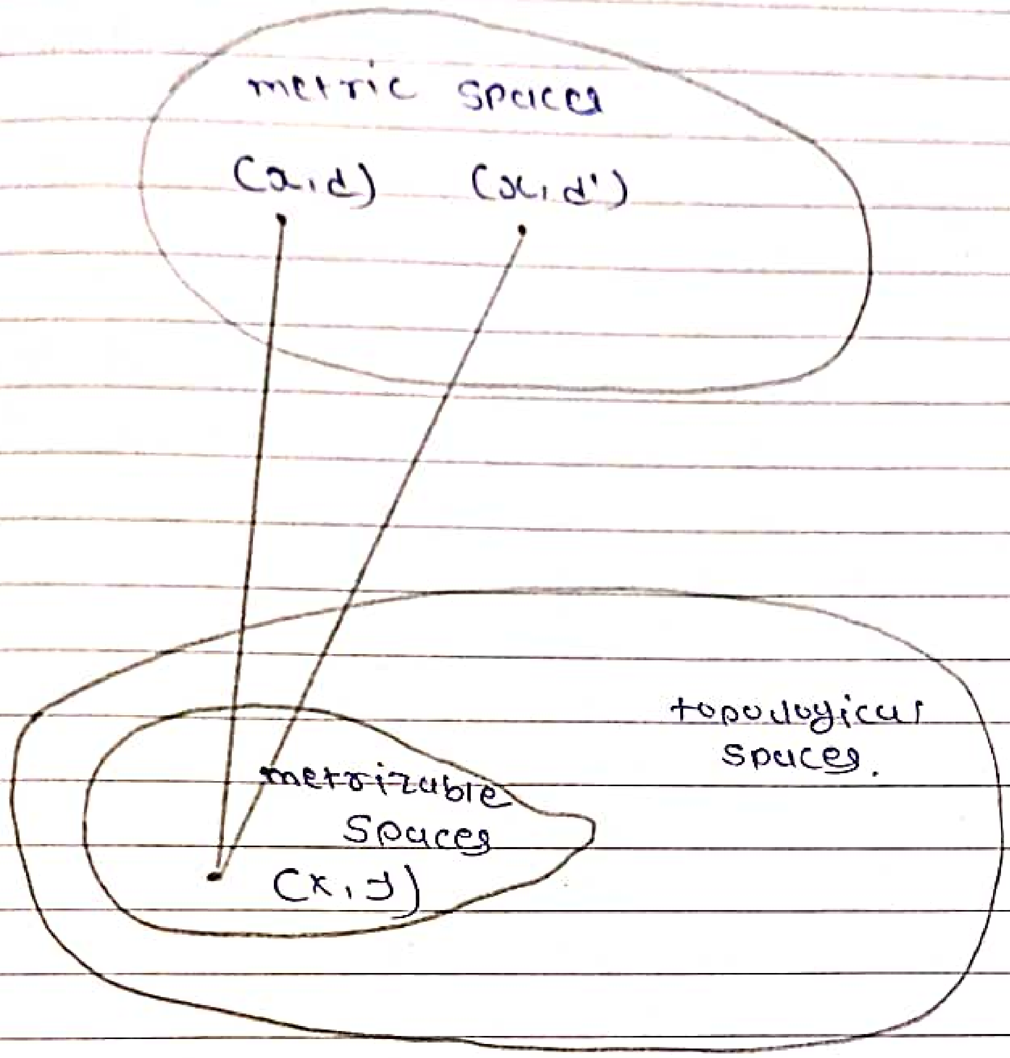
=> Then $c(\cup_{\alpha \in I} O_\alpha) = \cap_{\alpha \in I} C(O_\alpha)$. Either each of the sets, $c(O_\alpha) = X$, in which case the intersection is all of X, or at least subset of a finite set and hence finite.

Thus. $(X, \mathcal{T})$ is a topological space. The reader should verify that the remaining examples do, in fact, constitute examples of topological spaces.

=> Given a topological space $(X, \mathcal{T})$, the subsets O of X that belong to $\mathcal{T}$ are called 'open' sets. The adjective 'open' is used because, in the event that the topological space $(X, \mathcal{T})$ aries from a metric space (X, d), the subsets of X that are open in the associated topological space $(X, \mathcal{T})$ are precisely those subsets of X which are open in the metric space (X, d).

=> One may describe this situation by picturing the totality of topological spaces, ech each metric space (X, d) giving rise to its associated topological space $(X, \mathcal{T})$, as indicated in figure.

=> We shall see that two distinct metric space (X, d) and (X, d') may give rise to the same topogical space $(X, \mathcal{T})$. Also there are topological space $(Y, \mathcal{T}')$, such as Example above, which could not have arisen from a metric space.



⇒ In passing from a metric space to its associated topological space, we may say that the 'open' sets have been 'preserved'.

* Definition :-

Given a topological space $(X, \mathcal{J})$, a subsets N of X is called a neighbhood of a point $a \in X$ if N contains an open set that contains a.